PROJECT TUBEFLIGHT

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LINEARIZED FLOW PAST A FLAT PLATE
TR PT 6901

LINEARIZED FLOW PAST
A FLAT PLATE

by
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SUMMARY

This investigation presents a new technique for the approximate solution of incompressible boundary layer flow problems. Solutions are determined from linearized forms of both the Prandtl and the von Mises boundary layer equations. These forms are obtained by the introduction of a stretching function which provides a local linearization of the inertia term in the Prandtl equation, and of the viscous term in the von Mises equation. The linearized solution is derived as a functional dependent upon the stretching function. A first order differential equation is derived that determines this stretching function for a class of boundary layer problems characterized by a nonuniform initial velocity distribution, a constant freestream velocity, and a piecewise constant slip velocity at the wall. The approximate solutions obtained by introducing the stretching function into the linearized solution are compared directly with a previously unpublished exact solution for a simplified problem having a uniform initial velocity distribution.
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## LIST OF SYMBOLS

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<tr>
<td>$e(x)$</td>
<td>$e(x) = \hat{x}$, coordinate transformation for Prandtl stretching function</td>
</tr>
<tr>
<td>$g(x)$</td>
<td>$g(x) = \hat{x}$, coordinate transformation for von Mises stretching function</td>
</tr>
<tr>
<td>$k(x)$</td>
<td>von Mises stretching function</td>
</tr>
<tr>
<td>$k(x)$</td>
<td>Oseen stretching function</td>
</tr>
<tr>
<td>$\ell(x)$</td>
<td>reference length</td>
</tr>
<tr>
<td>$Re$</td>
<td>$Re = \frac{U_r \ell}{\nu}$, Reynolds number based on reference quantities</td>
</tr>
<tr>
<td>$u$</td>
<td>velocity parallel to wall</td>
</tr>
<tr>
<td>$\bar{u}$</td>
<td>velocity along a streamline</td>
</tr>
<tr>
<td>$\tilde{u}$</td>
<td>approximate solution to von Mises equation</td>
</tr>
<tr>
<td>$\hat{u}$</td>
<td>transformed approximate solution to von Mises equation</td>
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<tr>
<td>$\bar{u}$</td>
<td>approximate solution to Oseen equation</td>
</tr>
<tr>
<td>$\hat{u}$</td>
<td>transformed approximate solution to Oseen equation</td>
</tr>
<tr>
<td>$u_0(y)$</td>
<td>initial velocity distribution for Prandtl boundary layer equation</td>
</tr>
<tr>
<td>$\bar{u}_0(\psi)$</td>
<td>initial velocity distribution for von Mises boundary layer equation</td>
</tr>
<tr>
<td>$U$</td>
<td>constant slip velocity at the wall</td>
</tr>
<tr>
<td>$U_c$</td>
<td>constant velocity at outer edge of boundary layer</td>
</tr>
<tr>
<td>$U_r$</td>
<td>reference velocity</td>
</tr>
<tr>
<td>$x$</td>
<td>distance measured parallel to wall in physical plane</td>
</tr>
<tr>
<td>$\hat{x}$</td>
<td>$\hat{x} = \int^x_0 k(\rho) d\rho$, distance parallel to wall in transformed plane</td>
</tr>
<tr>
<td>$\check{x}$</td>
<td>$\check{x} = \int^x_0 \ell(\rho) d\rho$, distance parallel to wall in transformed plane</td>
</tr>
<tr>
<td>$y$</td>
<td>distance measured normal to wall</td>
</tr>
<tr>
<td>$\gamma(\hat{x})$</td>
<td>Oseen stretching function</td>
</tr>
<tr>
<td>$\varepsilon(\hat{x})$</td>
<td>von Mises stretching function</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density</td>
</tr>
<tr>
<td>$\nu$</td>
<td>kinematic viscosity</td>
</tr>
<tr>
<td>$\psi$</td>
<td>$\psi = \frac{\partial u}{\partial y}$, stream function</td>
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INTRODUCTION

Boundary layer flow has been under study for over 60 years, but the problems studied previously have primarily been those associated with uniform initial and boundary conditions. This paper is concerned with the solution of the steady two-dimensional laminar boundary layer (BL) equations for a more general type of initial and boundary conditions. This problem corresponds to an arbitrary nonuniform initial velocity (NUIV) distribution, a constant free stream velocity, and a piecewise constant slip velocity at the wall. Exact solutions are not in general available for this flow problem.

As a simplification of the NUIV case, we have the situation where there is a uniform initial velocity (UIV) distribution. Although the UIV problem can be solved exactly, only its two limiting cases appear to have received previous treatment in the literature. These are: zero slip velocity with constant free stream velocity (the classical Blasius problem) and constant slip velocity with zero free stream velocity (the moving wall problem).

In reference 1, it is suggested that an approximate solution could be obtained for the NUIV case by a new technique, which linearized the BL equations in a particular manner as its first step. For the Prandtl BL equations, the nonlinear inertia term is replaced with $\ell(x) u_x$, while for the von Mises BL equations the nonlinear viscous term is replaced by $k^{-1}(x) u \psi \psi$. Since the functions $\ell(x)$ and $k(x)$ are conveniently absorbed in a stretching transformation of the $x$ coordinate, they are denoted as the Oseen and von Mises stretching functions, respectively. The linearized solution is obtained as a functional involving the stretching functions.
The second step of this technique, the determination of $\lambda$ or $k$, is not developed in reference 1, and instead a primitive assumption is made to determine the value of the stretching function. The results derived by means of this assumption are promising, however, and indicate the value of determining the stretching function by rigorous means.

This paper develops a procedure to determine the stretching function for the NUIV case, and provides a check on the stretching function linearization (SFL) technique for the UIV case by comparing the SFL solution with the exact solution.
Before attempting to determine the stretching function, let us review two equivalent forms of the BL equations that describe the general flow problem shown in Figure 1.

The Prandtl BL equations, describing the steady two-dimensional, laminar motion of an incompressible fluid at constant velocity \( U_c \) parallel to a fixed, impermeable wall whose surface in motion at a constant velocity \( U \), are

\[
\begin{align*}
  u u_x + v u_y &= -y u y y \\
  u_x + v y &= 0
\end{align*}
\]  \hspace{1cm} (1a)

with the initial and boundary conditions

\[
\begin{align*}
  u(0, y) &= u_0(y) \\
  u(x, 0) &= U \\
  \lim_{y \to \infty} u(x, y) &= U_c
\end{align*}
\]  \hspace{1cm} (1b)

The change of variables with respect to a reference velocity \( U_r \) and a reference length \( l_r \) given by

\[
\begin{align*}
  u^* &= u/U_r \\
  v^* &= v/U_r \\
  u_o^* &= u_o/U_r \\
  x^* &= x/l_r \\
  U^* &= U/U_r \\
  U_c^* &= U_c/U_r \\
  y^* &= Re \cdot y/l_r \\
  Re &= U_r l_r / \nu
\end{align*}
\]  \hspace{1cm} (2)
renders equations 1a, b dimensionless and of the form

\[ u u_x + u u_y = u y y \]
\[ u_x + u y = 0 \]  
(3a)

\[ u(o, y) = u_o(y) \]
\[ u(x, o) = U \]
\[ u(x, y) = U_c \]  
(3b)

where we have finally dropped the *. The reference velocity is chosen
as follows: for \( U_c < U \), we choose \( U_r = U \), and for \( U_c > U \), we choose
\( U_r = U_c \). Equations 3a may be combined to give the single integro-
differential equation

\[ u u_x - u_y \int_o^y u_x d y = u y y \]  
(3c)

The change of variables

\[ \psi(x, y) = \int_o^y u(x, p) d p \]
\[ \bar{u}(x, \psi) = u(x, y) \]
\[ \bar{u}_o(\psi) = u_o(y) \]  
(4)

leads to the von Mises form of the problem

\[ \bar{u}^{-1} \bar{u}_x^2 = \bar{u}_y \psi \]  
(5a)

\[ \bar{u}(o, \psi) = \bar{u}_o(\psi) \]
\[ \bar{u}(x, o) = U \]
\[ \lim_{\psi \to \infty} \bar{u}(x, \psi) = U_c \]  
(5b)

Both of these problem formulations will be linearized as the first step
in obtaining an SFL solution for the NUIV case.

In the process of linearizing equation 3c, the nonlinear inertia
terms, \( uu_x - uy \int_0^y u_x \, dy \), are replaced by the single term \( l(x) u_x \). This linearization process should yield good results for problems in which the first nonlinear term dominates the second nonlinear term. However, if the second nonlinear term is dominant, such as in some NUIV problems, then correct results should not be expected. For this reason, the stretching function that linearizes the von Mises equation will be derived in detail, while only the results will be given for the stretching function that linearizes equation 3c.

von Mises Stretching Function

Let us obtain an approximate solution to equation 5 which satisfies the initial and boundary conditions exactly and the differential equation approximately. We may determine this solution, for which we use the symbol \( \tilde{u} \), from the linear equation

\[
k(x) \tilde{u}_x^2 = \tilde{u}^2 \psi
\]

subject to

\[
\tilde{u}(0, \psi) = \tilde{u}_o (\psi) \\
\tilde{u}(x, 0) = U \\
\lim_{\psi \to \infty} \tilde{u}(x, \psi) = U_c
\]

The multiplier \( k(x) \) is denoted as the stretching function for the von Mises form of the BL equations, or simply as the von Mises stretching function. The solution to equation 6 is easily obtained as a functional dependent upon \( k \). The approximate solution, denoted as the SFL solution, can be derived from this linearized solution only upon the determination of \( k \).

As a simple criterion to determine \( k(x) \), let us assume that
\( \overline{u} \) is just \( \overline{u} \) and then require that

\[
\int_{\psi}(\text{equation 5a - equation 6a}) \, d\psi = 0 \quad (7)
\]

This is equivalent to requiring that, at every \( x \), the rate of change of velocity along a streamline \( \overline{u}^{-1}u^2 \), and \( k(x) \overline{u}^2_x \) have the same value when integrated over all \( \psi \), i.e., that

\[
\int_{\psi}(\overline{u}^{-1}u^2_x - k(x)u^2_x) \, d\psi = 0 \quad (8)
\]

Of course, the quantity \( \overline{u} \) will not be point-wise correct but it should give good results for averaged quantities such as boundary layer thickness. The resulting \( k(x) \) is conveniently introduced in the equations in terms of the stretching transformation

\[
\frac{d}{d\hat{x}} = k(x) \frac{d}{dx} \quad (9a)
\]

and the change of variables

\[
\hat{x} = \int_{\rho}^x \frac{d\rho}{k(\rho)} = g(x) \quad (9b)
\]

\[
\hat{u}(x,\psi) = \hat{u}(\hat{x},\psi) = \hat{u}(g(x),\psi) \quad (9c)
\]

Equations 6a,b simplify to

\[
\hat{u}_x^2 = \hat{u}^2 \psi \quad (10a)
\]

\[
\hat{u}(0,\psi) = \overline{u}_o(\psi) \quad \hat{u}(\hat{x},0) = U \quad \lim_{\psi \to \infty} \hat{u}(\hat{x},\psi) = U_c \quad (10b)
\]

which is, of course, the classical heat equation.
Also, equation 8 may be transformed to

\[ \int_0^\infty (\hat{\mathbf{u}}^{-1} \hat{\mathbf{u}}_x^2 - k(x) \hat{\mathbf{u}}_x^2) \, d\psi = 0 \quad (11) \]

or alternatively

\[ k(x) = \frac{\int_0^\infty \hat{\mathbf{u}}^{-1} \hat{\mathbf{u}}_x^2 \, d\psi}{\int_0^\infty \hat{\mathbf{u}}_x^2 \, d\psi} \quad (12) \]

For \( \hat{\mathbf{u}} \) positive, \( k \) will be positive and the transformation \( g \) will be monotonic. If the solution of equations 10a,b in terms of the stretched \( x \) coordinate is substituted into the expression on the right hand side of equation 10, and the integration is performed, then it is easily seen that the resulting expression will be a function of \( \hat{x} \). If this function of \( \hat{x} \) is denoted as \( \varepsilon (\hat{x}) \), then we have

\[ k(x) = \varepsilon (\hat{x}) \quad (13) \]

The stretching function may be determined by substituting equation 9b in this result to get

\[ k(x) = \varepsilon (\mathcal{g}(x)) \quad (14a) \]

or

\[ k(x) = \varepsilon \left( \int_0^x \frac{d\rho}{k(\rho)} \right) \quad (14b) \]

A valuable result concerning \( k \) may be obtained by noting that \( g' \) is just \( 1/k \) and substituting this result in equation 14a to yield

\[ g' = \frac{1}{\varepsilon (\mathcal{g}(x))} \]

This first order nonlinear ordinary differential equation will yield a unique solution for \( g \) provided \( 1/\varepsilon \) is continuous and satisfies the
Lipschitz condition. Thus for \( g' \neq 0 \), any \( k(x) \) determined as a solution of equation 14b, numerically or otherwise, will be unique. With this result, the solution of the BL equation may be approximated by the successive solution of a second order linear partial differential equation which determines the functional form of \( \varepsilon \), and a first order non-linear ordinary differential equation which determines \( k \).

Thus, the approximate solution to the general boundary layer flow problem given in equation 5 may be determined by the following steps: First, the system given by equation 10a,b is solved for \( \hat{u} \). A solution to this system for a large class of initial velocity distributions is given in Appendix I. Next, this result is substituted in equation 12 to get the functional form for \( \varepsilon (\hat{x}) \). Third, the stretching function is determined from equation 14b. Fourth and last, the velocity \( \hat{u} \) is determined from equation 9c.

There are only two possibilities for the functional form of \( \varepsilon (\hat{x}) \), either it is a function of \( \hat{x} \) or it is a constant. The first possibility occurs for the NUIV case and requires that an iterative process be used to determine \( k(x) \). One such process might be to choose \( k(x) = k^{(1)}(x) \) and then to calculate successively higher approximations for \( k(x) \), i.e., \( k^{(n)}(x) \), through repeated use of the expression

\[
k^{(n+1)}(x) = \varepsilon \left( \int_{\hat{x}}^{x} \frac{d\rho}{k^{(n)}(\rho)} \right)
\]

until there is the desired agreement between \( k^{(n+1)}(x) \) and \( k^{(n)}(x) \).

For the UIV case, the resulting velocity distributions are affine and it is to be expected that both \( \varepsilon (\hat{x}) \) and \( k(x) \) will be constants. Indeed, if the solution for the UIV case, which is given by
equation 1-1 with \( p = m = 1 \) and \( U_1 = U \), is substituted in equation 12, then the result may be given by

\[
k(x) = 2 \int_{0}^{\infty} \left( U^2 + (U_c - U^2) \text{erf} \, p \right) \exp(-p^2) \, dp
\]  

(16)

The stretching function has been determined from equation 16 by numerical quadrature and is shown in Figure 2a for \( U = 1 \), \( 0 \leq U_c \leq 1 \) and in Figure 2b for \( U_c = 1 \), \( 0 \leq U \leq 1 \).

**Oseen Stretching Function**

Equation 3c can also be linearized by a stretching function approach. Let us obtain an approximate solution to this equation which satisfies the initial and boundary conditions exactly and the differential equation approximately. We may determine this solution, for which we use the symbol \( \bar{u} \), from the linear equation

\[
\ell(x) \ddot{\bar{u}}_x = \bar{u}_{yy}
\]  

(17)

Because of the similarity of this equation to the Oseen equation, the stretching function \( \ell(x) \) is denoted the Oseen stretching function. Proceeding in a similar fashion as for the von Mises equation, the stretching transformation

\[
\frac{d}{dx} = \ell(x) \frac{d}{dx}
\]  

(18a)

and the change of variables

\[
\tilde{x} = \int_{0}^{x} \frac{dp}{\ell(p)} = e(x)
\]  

(18b)

\[
\tilde{u}(x,y) = \bar{u}(\tilde{x},y) = \bar{u}(e(x),y)
\]  

(18c)
simplifies equation 17 to
\[ \ddot{u}_x = \ddot{u}_{yy} \]
(19a)

and the initial and boundary conditions of equation 3b to
\[ \begin{align*}
\dot{u}(0,y) &= u_0(y) \\
\dot{u}(\bar{x},0) &= \bar{U} \\
\lim_{y \to \infty} \ddot{u}(\bar{x},y) &= U_c
\end{align*} \]
(19b)

For the Prandtl equation, the criterion for the determination of \( \lambda(x) \) would be that
\[ \int_0^\infty (\ddot{u}_x - \ddot{u}_y \int_0^y \ddot{u}_x \, dy - \lambda(x) \ddot{u}_x) \, dy = 0 \]
(20)
or that
\[ \lambda(x) = \frac{\int_0^\infty (\ddot{u}_x - \ddot{u}_y \int_0^y \ddot{u}_x \, dy) \, dy}{\int_0^\infty \ddot{u}_x \, dy} \]
(21)

Unfortunately, even for positive \( \ddot{u} \), \( \lambda(x) \) may pass through zero making e non-monotonic which invalidates the solution. If the solution of equation 19 is substituted in the right hand side of equation 21, and the integration is performed, then the result will be a function of \( \bar{x} \). If this function of \( \bar{x} \) is denoted as \( \gamma(\bar{x}) \), we have then
\[ \lambda(x) = \gamma(\bar{x}) \]
(22)

or
\[ \lambda(x) = \gamma(e(x)) \]
(23a)

or
\[ \lambda(x) = \gamma(\int_0^x \frac{dp}{E(p)}) \]
(23b)
Thus, the SFL solution of the Prandtl form of the BL equations can be determined in the same manner as the SFL solution of the von Mises form of the BL equations. For the UIV case, the analogous result to equation 16 may be integrated exactly to yield

\[ \ell(x) = (2 - \sqrt{2})U - (1 - \sqrt{2})U_c \]  

(24)
RESULTS AND DISCUSSION

Now that a method has been formulated for the determination of \( k(x) \) and \( \ell(x) \), let us examine the error in the approximate solution of the BL equations obtained by the SFL technique. This will be accomplished for the UIV case by comparing the SFL solution directly to the exact solution. The exact solution for the UIV case is contained in Appendix II.

Figure 3a, b compares the shear stress at the wall calculated from the exact and SFL solution of the BL equations. For the Oseen stretching function approach, the absolute relative error is less than 10%, while for the von Mises stretching function it is less than 14%. The maximum error in each case occurs at the limiting cases of \( U = 0, U_c = 1 \), and \( U = 1, U_c = 0 \). By comparison, the approximate solution due to Oseen is in error by 76% for the case where \( U = 0, U_c = 1 \). For the range given by \( U = 1.0, 0.3 < U_c < 1.0 \) and \( U_c = 1.0, 0.3 < U < 1.0 \), there is less than 3% error in the shear stress.

Shown in Figure 4a,b and c is a comparison of the velocity distributions determined from the SFL and the exact solutions of the BL equations. The general agreement for all three examples is good. Based on the shear stress results, this agreement for the example with \( U = 1.0, U_c = 0.5 \) should be representative of the error in the SFL solution for \( U = 1.0, 0.3 < U_c < 1.0 \) and \( U_c = 1.0, 0.3 < U < 1.0 \). Since the stretching function for the UIV case simply alters the scale of the \( y \) coordinate, i.e., \( \hat{u}(\hat{x},y) = u(x, \sqrt{ky}) \), the agreement between the exact and the SFL solutions is not easily improved. The SFL technique apparently yields a result that is a compromise between a correct shear and a correct BL thickness. For the \( U = 1, U_c = 0 \) case, for instance, a
larger stretching function would yield a more correct BL thickness and a less correct wall shear stress, while a smaller stretching function would yield the opposite result. A similar conclusion can also be reached for the $U = 0, U_c = 1$ case.

Based on these results, it appears that the technique developed for the determination of the stretching function is valid and that the SFL technique is a good means for obtaining an approximate solution to the BL equations, at least for the UIV case. A check of the validity of the SFL technique for the NUIV case, and its application to predict the BL flow in a circular cavity will be treated in a future paper.
APPENDIX I

SOLUTIONS FOR A GENERAL CLASS OF BOUNDARY LAYER PROBLEMS

Let us consider the class of incompressible steady two-
dimensional laminar boundary layers produced by the parallel motion
at constant velocity of \( m \) discrete segments of the surface on a fixed
flat plate aligned with a uniform free stream of constant velocity \( U_c \).
(All of the velocities described in this appendix are actually velocity
ratios). The \( p^{th} \) segment \( (p \leq m) \), of length \( a_p \), has a surface velocity
\( U_p \) which may be considered as produced by a moving belt. Utilizing the
linearized von Mises equation, (i.e. equation 10) and the results of
Appendix I of reference 1, we find that the velocity distribution at a
distance \( \hat{x}_p \) downstream from the beginning of the \( p^{th} \) segment is given
by

\[
\hat{u}_p^2 (\hat{x}_p, \psi) = U_c^2 \sigma (\hat{x}_p + \sum_{l=1}^{p-1} \hat{u}_l) + \sum_{n=1}^{p} U_n^2 \left[ \sigma (\hat{x}_p + \sum_{l=n+1}^{p-1} \hat{u}_l) - \sigma (\hat{x}_p) \right]
\]

(1-1)

where

\[
k_l (x) = \int_{-\infty}^{x} \frac{\hat{u}_l \hat{u}_l}{k_l (p)} \frac{d \psi}{d \psi}
\]

\[
\hat{a}_i = \int_a^b \frac{d \psi}{k_l (p)}
\]

\[
\sigma (\alpha) = \text{erf} (\frac{\alpha}{\sqrt{2}})
\]

\[
\sigma (\hat{x}_p + \sum_{l=p+1}^{p-1} \hat{a}_l) = 1
\]

and \( u_p (x_p, y) \) may be recovered by the subsequent use of equations 9c and 4.

In terms of this notation, the velocity distribution for the
Blasius problem corresponds to \( U_c > 0 \), \( m = 1 \), \( U_1 = 0 \), while for the moving
wall problem it corresponds to \( U_c = 0 \), \( m = 1 \), \( U_1 = U \). The velocity dis-
tribution \( u_p (x_p, y) \) determined from equations 1-1, 9 and 4 may in general
have more than one inflection point and a local velocity within the BL which may exceed the free stream velocity or the surface velocity.

Although the result given by Equation I-1 is interesting of itself, it can be put to a more practical use. If it is desired to determine the development of a given NUIV distribution on either a stationary or moving wall, one simply determines the set of $a_n$ and $U_n$ such that $u_{n+1}(0, \gamma)$ matches the given profile to the desired accuracy. The number of $a_n$ and $U_n$ required to match the given profile will depend on its complexity. In any case, the velocity of the stationary or moving wall will be $U_{n+1}$. The substitution of these values of $a_n$ and $U_n$ in equation I-1 yields the desired BL velocity development, $u_{n+1}$, in the $\hat{x}$ plane. The subsequent use of equations 9c and 4 will then transform this result to the physical plane.
APPENDIX II

EXACT SOLUTION OF THE BOUNDARY LAYER EQUATIONS FOR THE UIV CASE

The exact solution of equation 1 with $u_o(y) = U_c$ is obtained by the application of the similarity transform

$$
\psi = \sqrt{\nu x} \quad U_r^{\frac{1}{2}} f(\eta) \\
\eta = \frac{y U_r^{\frac{1}{2}}}{\sqrt{\nu x}}
$$

which simplifies the problem to an ordinary differential equation of the form

$$
2f'' + f'f = 0
$$

with the boundary conditions

$$
f(0) = 0 \\
f'(0) = U \\
\lim_{\eta \to \infty} f'(\eta) = U_c
$$

The solution of this system has been obtained numerically for $U \geq 0$, $U_c \geq 0$ by the use of a modified Hamming's predictor-corrector technique. The results agree for the limiting cases of $U = 0$, $U_c = 1$ and $U = 1$, $U_c = 0$ with those of reference 2 and reference 4, respectively.
REFERENCES


Fig. 2a
von Mises Stretching Function
for $U \gg U_c$

Fig. 2b
von Mises Stretching Function
for $U_c \gg U$
Fig. 3a
Wall Shear Stress for $U \geq U_c$
Fig. 3b
Wall Shear Stress for $U_c \gg U$
SFL Boundary Layer Velocity Profiles for $U = 1, U_c = 0$
**Fig. 4b**

SFL Boundary Layer Velocity Profiles for $U = 1$, $U_c = .5$
Fig. 4c
SFL Boundary Layer Velocity Profiles for $U = 0$, $U_c = 1$